On positivity and decay of solutions of second order elliptic equations on Riemannian manifolds

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0. Introduction.

Dedicated to the memory of Professor
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It is well known that positivity properties appear in various forms in the study of second order elliptic equations. Thus for instance a number of authors have discussed the connection between the existence of positive solutions in a neighborhood of infinity and the nature of the spectrum (see Allegretto [2 ,3 ,4], Piepenbrink [9,10], Moss-Piepenbrink [7]). A rather simple result of this type is the following. Consider the Schrödinger operator $P = -\Delta + q$ on $\mathbb{R}^n$. Then for a general class of real functions q the operator $P = -\Delta + q$ has a non-negative spectrum (in the sense that

$$\int_{\mathbb{R}^n} (|\nabla\varphi|^2 + q\,|\varphi|^2)dx \geq 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$) if and only if there exists a positive solution of the equation $Pu = 0$ in $\mathbb{R}^n$.

In this paper we propose to discuss in a systematic way various closely connected positivity properties of second order elliptic operators defined on Riemannian manifolds of which the result described above is a very special

2. Energy space $\mathcal{D}^1_V(\Omega)$.

AAP-principle: Assume $\exists\, u_* > 0$:

$$-\Delta u_* + V(x) u_* \geq f \text{ in } \Omega \text{ for some } f \geq 0.$$

Then $E_V(\varphi) \geq \int_\Omega \frac{f}{u_*} \varphi^2 \quad \forall \varphi \in C_c^\infty(\Omega)$.

Example ($-\Delta$ on $\mathbb{R}^2$). Take $u_* = (\log|x|)^{\frac{1}{2}}$

$$-\Delta u_* = \frac{1}{4} \frac{1}{|x|^2 (\log|x|)^{3/2}} = f, \quad \Omega = \mathbb{R}^2 \setminus \bar{B}_1.$$

$$\int_\Omega |\nabla \varphi|^2 \geq \int_\Omega \underbrace{\frac{f}{u_*}}\, \varphi^2 = \int_\Omega \frac{1}{4} \frac{\varphi^2}{|x|^2 (\log|x|)^2}$$

$$\forall \varphi \in C_c^\infty(\Omega)$$

Theorem. Assume $\exists\, \alpha \in L^1_{loc}(\Omega)$, $\alpha > 0$ a.e.:

$$(*) \quad E_V(\varphi) \geq \int_\Omega \underbrace{\alpha(x)}_{\leq \frac{S}{u_*}} \varphi^2\, dx \quad \forall \varphi \in C_c^\infty(\Omega).$$

Then the completion of $C_c^\infty(\Omega)$ w.r.t.

$\|\varphi\|_V = (E_V(\varphi))^{1/2}$ is the Hilbert spac $\mathcal{D}^1_V(\Omega)$,

$\langle \varphi, \psi \rangle_V = \int_\Omega \nabla\varphi \nabla\psi + \int_\Omega V\varphi\psi$ – scalar prod.,

$$\mathcal{D}^1_V(\Omega) \hookrightarrow L^2(\Omega, \alpha(x)\,dx).$$

$(*)$ – Lambda property.

1) $\|\varphi\|_V$ is a norm on $C_c^\infty(\Omega)$ $\left(\|\varphi\|_V = 0 \Leftrightarrow \varphi = 0 \right)$

2) (φ_n) is a Cauchy sequence w.r.t. to $\|\varphi\|_V$

$\Longrightarrow$ (φ_n) is a Cauchy seq. in $L^2(\Omega, \alpha(x)dx)$

$\Longrightarrow$ $\varphi_n \xrightarrow[a.e.]{} \varphi$

$\Longrightarrow$ $E_V(\varphi) := \lim\limits_{n \to \infty} E(\varphi_n)$,

$\varphi \in \mathcal{D}_V^1(\Omega) \subset L(\Omega, \alpha(x)dx)$

$$\left(\int_\Omega \underbrace{|\varphi_n - \varphi_m|^2 \alpha(x)\, dx}_{\to 0} \right)^{1/2} \leq \left(E(\varphi_n - \varphi_m) \right)^{1/2} \to 0$$

Examples:

1) $-\Delta$ on $\mathbb{R}^N$, $N \geq 3$.

Take $u_* = (1+|x|^2)^{-\frac{N-2}{2}}$ — Talenti function

$$-\Delta u_* = c\,(1+|x|^2)^{-\frac{N+2}{2}} \simeq u_*^{\frac{N+2}{N-2}} \text{ in } \mathbb{R}^N$$

$$\alpha(x) = \frac{f}{u_*} \simeq \frac{u_*^{\frac{N+2}{N-2}}}{u_*} \simeq u_*^{\frac{4}{N-2}} \simeq (1+|x|^2)^{-1}$$

$$\Rightarrow \mathcal{D}^1_0(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N, (1+|x|^2)^{-1}dx)$$

$\mathcal{D}^1_0(\mathbb{R}^N)$ – completion of $C_0^\infty(\mathbb{R}^N)$ w.r.t. $\left(\int |\nabla u|^2\right)^{1/2}$

$\mathcal{D}^1_0(\mathbb{R}^N) \subset L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, $\mathcal{D}^1_0(\mathbb{R}^N) \not\subset L^2(\mathbb{R}^N)$

$\sim \mathcal{D}^1_0(\mathbb{R}^N) \neq H^1(\mathbb{R}^N)$!

2) $-\Delta$ on $\mathbb{R}^2$ – does not satisfy Ω-property
(const is the only positive superharmonic !)

$\Rightarrow \mathcal{D}^1_0(\mathbb{R}^2)$ is not well defined !

3) $-\Delta$ on $\mathbb{R}^2 \setminus \overline{B_1}$

Take $u_* = (\log|x|)^{1/2} > 0$ in $\mathbb{R}^2 \setminus \overline{B_1}$

$$\lambda(x) = \frac{f}{u_*} = \frac{1}{4} \frac{1}{|x|^2 (\log|x|)^2} .$$

$$\mathcal{D}^1_0(\mathbb{R}^2 \setminus B_1) \subset L^2(\mathbb{R}^2 \setminus B_1, \lambda(x)\,dx).$$

Remark: Assume $E_V(\varphi) \geq 0 \ \forall \varphi \in C_0^\infty(\Omega)$ but E_V does not satisfy α-property.

Then $-\Delta + V$ has just one (up to scalar) positive supersolution = solution and $-\Delta + V$ is called critical operator.

E_V satisfies α-property $\Longrightarrow$ $-\Delta + V$ is subcritical operator

[Pinchover, Tintarev, JFA 2006]

Examples: 1) $-\Delta$ on $\mathbb{R}^2$ — critical

2) Ω - bounded, λ_1 — 1st Dirichlet eigenvalue

$-\Delta - \lambda_1$ is critical on Ω

(φ_1 is the only positive solution!)

$$\int |\nabla \varphi|^2 \geq \lambda_1 \int \varphi^2 \quad \forall \varphi \in C_c^\infty(\Omega)$$

$$(-\Delta - \lambda_1)\varphi_1 = 0$$

Theorem. $\forall f \in \left(\mathcal{D}'_V(\Omega)\right)^* \quad \exists\, u_f \in \mathcal{D}'_V(\Omega),$

$$-\Delta u_f + V u_f = f \text{ in } \Omega.$$

◀ Minimize the functional

$$\underbrace{\frac{1}{2}\|\varphi\|_V^2}_{\frac{1}{2}E_V} - \langle f, \varphi \rangle \longrightarrow \min \text{ on } \mathcal{D}'_V(\Omega)$$

▶

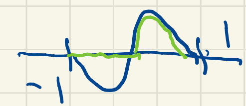
Theorem (weak Maximum Principle)

Assume $u \in \mathcal{D}^1_V(\Omega)$ is a supersolution, i.e.

(*) $-\Delta u + Vu = f \geq 0$ in Ω, for some $f \in (\mathcal{D}^1_V(\Omega))^*$

Then $u \geq 0$.

◀ $u = u^+ - u^-$, $\quad u^+, u^- \in \mathcal{D}^1_V(\Omega)$.



$\varphi = u^-$

$$0 \leq \int \nabla u \nabla \varphi + \int V u \varphi = \int (\nabla u^+ - \nabla u^-) \nabla u^- + \int V (u^+ - u^-) u^-$$

$$= -\int |\nabla u^-|^2 - \int V (u^-)^2 = - \underbrace{E_V(u^-)}_{\geq 0} \leq 0$$

$$\Longrightarrow \quad E_V(u^-) = 0 \quad \Longrightarrow \quad u^- = 0$$ ▶

Corollary (Weak comparison principle)

Let $u, v \in H^1_{loc}(\Omega) \cap L^1_{loc}(\Omega, V(x)dx)$ are sub and supersolutions:

$$-\Delta u + V(x)u \geq 0, \quad -\Delta v + V(x)v \leq 0 \text{ in } \Omega$$

and $(u-v)^- \in \mathcal{D}^1_V(\Omega)$. $\simeq u \geq v$ on $\partial\Omega$

Then $u \geq v$ in Ω.

◀ $-\Delta(u-v) + V(x)(u-v) \geq 0$ in Ω.

Take $\varphi_n \in C_c^\infty(\Omega)$, $0 \leq \varphi_n \to (u-v)^- \in \mathcal{D}^1_V$, repeat previous argument ▶

Three possibilities for $-\Delta+V$ in Ω:

1) $\exists \varphi \neq 0$: $E_V(\varphi) < 0 \Rightarrow$ no positive supersol

2) $E_V(\varphi) \geq 0$ and $-\Delta+V$ is critical

$\Rightarrow$ exactly one positive (super)solution

3) $E_V(\varphi) \geq 0$ ⊕ α-property ($-\Delta+V$ is subcritical

$\Rightarrow$ large cone of positive supersolutions

⊕ variational principle in $\mathcal{D}_V^1(\Omega)$

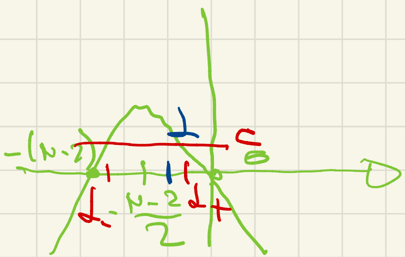
⊕ comparison principle

Exercise. (Hardy operator $-\Delta - \frac{c}{|x|^2}$ in $\mathbb{R}^N$)

Show that $-\Delta - \frac{c}{|x|^2}$ satisfies λ-property in $\mathbb{R}^N$

$\forall\ c \in (0, c_H)$, $c_H = \left(\frac{N-2}{2}\right)^2$ and $N \geq 3$.

◭ $u_* = |x|^\alpha$ $\quad -\Delta u_* - \frac{c}{|x|^2} u_* = \big(\underbrace{-\alpha(\alpha+N-2)-c}_{>0}\big)|x|^{\alpha-2}$



$\alpha_- < \alpha_+$ – roots of $-\alpha(\alpha+N-2) = c$

$\alpha \in \left(-\frac{N-2}{2}, \alpha_+\right)$

$|x|^\alpha \in H^1_{loc}(\mathbb{R}^N)$

$$-\Delta u + \frac{c}{|x|^2} u = \underbrace{c_d}_{>0} |x|^{d-2}$$

$$a(x) = \frac{f}{u} = \frac{c_d}{|x|^2}$$

$$\mathcal{D}^1_{\frac{c}{|x|^2}}(\mathbb{R}^N) \subset L^2\left(\mathbb{R}^N, \frac{c_d}{|x|^2}\,dx\right),$$

$$\forall c < C_H \quad \forall N \geq 3.$$